

# FLAT FAMILIES BY STRONGLY STABLE IDEALS AND A GENERALIZATION OF GRÖBNER BASES

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**ABSTRACT.** Let  $J$  be a strongly stable monomial ideal in  $S = K[x_1, \dots, x_n]$  and let  $\mathcal{Mf}(J)$  be the family of all homogeneous ideals  $I$  in  $S$  such that the set of all terms outside  $J$  is a  $K$ -vector basis of the quotient  $S/I$ . We show that an ideal  $I$  belongs to  $\mathcal{Mf}(J)$  if and only if it is generated by a special set of polynomials, the  $J$ -marked basis of  $I$ , that in some sense generalizes the notion of reduced Gröbner basis and its constructive capabilities. Indeed, although not every  $J$ -marked basis is a Gröbner basis with respect to some term order, a sort of normal form modulo  $I \in \mathcal{Mf}(J)$  can be computed for every homogeneous polynomial, so that a  $J$ -marked basis can be characterized by a Buchberger-like criterion. Using  $J$ -marked bases, we prove that the family  $\mathcal{Mf}(J)$  can be endowed, in a very natural way, with a structure of affine scheme that turns out to be homogeneous with respect to a non-standard grading and flat in the origin (the point corresponding to  $J$ ), thanks to properties of  $J$ -marked bases analogous to those of Gröbner bases about syzygies.

## INTRODUCTION

Let  $J$  be any monomial ideal in the polynomial ring  $S := K[x_0, \dots, x_n]$  in  $n + 1$  variables endowed so that  $x_0 < x_1 < \dots < x_n$  and let us denote  $\mathcal{N}(J)$  the set of terms outside  $J$ . In this paper we consider the family  $\mathcal{Mf}(J)$  of ideals  $I$  of  $S$  such that  $S = I \oplus \langle \mathcal{N}(J) \rangle$  as a  $K$ -vector space and investigate under which conditions this family is in some natural way an algebraic scheme. If  $\mathcal{N}(J)$  is not finite, the family of such ideals can be too large. For instance, if  $J = (x_0) \subset K[x_0, x_1]$ , the family of all ideals such that  $S/I$  is generated by  $\mathcal{N}(J) = \{x_1^n : n \in \mathbb{N}\}$  depends on infinitely many parameters. For this reason we restrict ourselves to the homogeneous case.

To study the family  $\mathcal{Mf}(J)$  we introduce a set of particular homogeneous polynomials, called  $J$ -marked set, that becomes a  $J$ -marked basis when it generates an ideal  $I$  that belongs to  $\mathcal{Mf}(J)$ . If  $J$  is strongly stable a  $J$ -marked basis satisfies most of the good properties of a reduced homogeneous Gröbner basis and, for this reason, we assume that  $J$  is strongly stable. However, even under this assumption, a  $J$ -marked basis does not need to be a Gröbner basis (Example 3.18). We show that a suitable rewriting procedure allows to compute a sort of normal forms and to recognize a  $J$ -marked basis by a Buchberger-like criterion. This criterion is the tool by which we construct the family  $\mathcal{Mf}(J)$  following the line of the computation of a Gröbner stratum, that is the family of all ideals that have  $J$  as initial ideal with respect to a fixed term order. In the last years, several authors have been working on Gröbner strata, proving that they have a natural and well defined structure of algebraic schemes, that springs out of a procedure based on Buchberger's algorithm [4, 11, 16, 18, 19], and that they are homogeneous with respect to a non standard positive grading over  $\mathbb{Z}^{n+1}$  [6]. In this context, it is worth also

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to recall that in [13] a method is described to compute all liftings of a homogeneous ideal with an approach different from, but close to the method applied to study Gröbner strata.

The paper is organized in the following way. In section 1 we give definitions and basic properties of  $J$ -marked sets and bases, with several examples. In section 2, in the hypothesis that  $J$  is strongly stable, we prove the existence of a sort of normal form, modulo the ideal generated by a  $J$ -marked set, for every homogeneous polynomial (Theorem 2.2). A consequence is that, if  $J$  is strongly stable, a  $J$ -marked set  $G$  is a  $J$ -marked basis if and only if  $J$  and the ideal generated by  $G$  share the same Hilbert function (Corollaries 2.3 and 2.4). From now we suppose that  $J$  is strongly stable and in section 3 define a total order (Definitions 3.4 and 3.9) on some special polynomials and give an algorithm to compute our normal forms by a rewriting procedure. This computation opens the access to effective methods for  $J$ -marked bases, as a Buchberger-like criterion (Theorem 3.12) that recognizes when a  $J$ -marked set is a  $J$ -marked basis  $G$ , also allowing to lift syzygies of  $J$  to syzygies of  $G$ .

In section 4 we study the family  $\mathcal{Mf}(J)$ , computing it by the Buchberger-like criterion and showing that there is a bijective correspondence between the ideals of  $\mathcal{Mf}(J)$  and the points of an affine scheme (Theorem 4.1). A possible objection to our construction is that it depends on a procedure of reduction, which is not unique in general. For this reason we show that  $\mathcal{Mf}(J)$  has a structure of an affine scheme, that is given by the ideal generated by minors of some matrices and that is homogeneous with respect to a non-standard grading over the additive group  $\mathbb{Z}^{n+1}$  (Lemma 4.2 and Theorem 4.4). Moreover, we note that  $\mathcal{Mf}(J)$  is flat in  $J$  and that the Castelnuovo-Mumford regularity of every ideal  $I \in \mathcal{Mf}(J)$  is bounded from above by the Castelnuovo-Mumford regularity of  $J$  (Proposition 4.5). In the Appendix, over a field  $K$  of characteristic zero, we give an explicit computation of a family  $\mathcal{Mf}(J)$  which is scheme-theoretically isomorphic to a locally closed subset of the Hilbert scheme of 8 points in  $\mathbb{P}^2$  (see also [12]). We note that it strictly contains the union of all Gröbner strata with  $J$  as initial ideal and that it is not isomorphic to an affine space, even though the point corresponding to  $J$  is smooth.

We refer to [3, 10, 14] for definitions and results about Gröbner bases and to [20] for definitions and results about Hilbert functions of standard graded algebras.

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## 1. GENERATORS OF A QUOTIENT $S/I$ AND GENERATORS OF $I$

In this section we investigate relations among generators of a homogeneous ideal  $I$  of  $S$  and generators of the quotient  $S/I$ , under some fixed conditions on generators of  $S/I$ .

For every integer  $m \geq 0$ , the  $K$ -vector space of all  $m$ -degree homogeneous polynomials of  $I$  is denoted  $I_m$ . The *initial degree* of an ideal  $I$  is the integer  $\alpha_I := \min\{m \in \mathbb{N} : I_m \neq 0\}$ .

We will denote by  $x^\alpha = x_0^{\alpha_0} \dots x_n^{\alpha_n}$  any term in  $S$ ,  $|\alpha|$  is its degree, and we say that  $x^\alpha$  divides  $x^\beta$  (for short  $x^\alpha | x^\beta$ ) if there exists a term  $x^\gamma$  such that  $x^\beta = x^\alpha x^\gamma$ . For every term  $x^\alpha \neq 1$  we set  $\min(x^\alpha) = \min\{x_i : x_i | x^\alpha\}$  and  $\max(x^\alpha) = \max\{x_i : x_i | x^\alpha\}$ .

**Definition 1.1.** The support  $\text{Supp}(h)$  of a polynomial  $h$  is the set of terms that occur in  $h$  with non-null coefficients.

If  $J$  is a monomial ideal,  $B_J$  denotes its (minimal) monomial basis and  $\mathcal{N}(J)$  its *sous-escalier*, that is the set of terms outside  $J$ . For every polynomial  $f$  of  $J$ , we get  $\text{Supp}(f) \cap \mathcal{N}(J) = \emptyset$ .

**Definition 1.2.** Given a monomial ideal  $J$  and an ideal  $I$ , a  $J$ -normal form modulo  $I$  of a polynomial  $h$  is a polynomial  $h_0$  such that  $h - h_0 \in I$  and  $\text{Supp}(h_0) \subseteq \mathcal{N}(J)$ .

If  $I$  is homogeneous, the  $J$ -normal form modulo  $I$  of a homogeneous polynomial  $h$  is supposed to be homogeneous too.

**Definition 1.3.** [17] A *marked polynomial* is a polynomial  $f \in S$  together with a specified term of  $\text{Supp}(f)$  that will be called *head term* of  $f$  and denoted  $Ht(f)$ .

**Definition 1.4.** A finite set  $G$  of homogeneous marked polynomials  $f_\alpha = x^\alpha - \sum c_{\alpha\gamma}x^\gamma$ , with  $Ht(f_\alpha) = x^\alpha$ , is called  $J$ -marked set if the head terms  $Ht(f_\alpha)$  (different two by two) form the monomial basis  $B_J$  of a monomial ideal  $J$  and every  $x^\gamma$  belongs to  $\mathcal{N}(J)$ , so that  $|\text{Supp}(f) \cap J| = 1$ . A  $J$ -marked set  $G$  is a  $J$ -marked basis if  $\mathcal{N}(J)$  is a basis of  $S/(G)$  as a  $K$ -vector space, i.e.  $S = (G) \oplus \langle \mathcal{N}(J) \rangle$  as a  $K$ -vector space.

*Remark 1.5.* The ideal  $(G)$  generated by a  $J$ -marked basis  $G$  has the same Hilbert function of  $J$ , hence  $\dim_K J_m = \dim_K (G)_m$  for every  $m \geq 0$ , by the definition of  $J$ -marked basis itself.

**Definition 1.6.** The family of all homogeneous ideals  $I$  such that  $\mathcal{N}(J)$  is a basis of the quotient  $S/I$  as a  $K$ -vector space will be denoted  $\mathcal{Mf}(J)$  and called  $J$ -marked family.

*Remark 1.7.* (1) If  $I$  belongs to  $\mathcal{Mf}(J)$ , then  $I$  contains a  $J$ -marked set.

(2) A  $J$ -marked family  $\mathcal{Mf}(J)$  contains every homogeneous ideal having  $J$  as initial ideal with respect to some term order, but it can also contain other ideals, as we will see in Example 3.18.

**Proposition 1.8.** Let  $G$  be a  $J$ -marked set. The following facts are equivalent:

- (i)  $G$  is a  $J$ -marked basis;
- (ii) the ideal  $(G)$  belongs to  $\mathcal{Mf}(J)$ ;
- (iii) every polynomial  $h$  of  $S$  has a unique  $J$ -normal form modulo  $(G)$ .

*Proof.* It follows by the definition of  $J$ -marked basis. □

*Remark 1.9.* A  $J$ -marked basis is unique for the ideal that it generates, by the unicity of  $B_J$  and of the  $J$ -normal forms of monomials.

In next examples we will see that not every  $J$ -marked set  $G$  is also a  $J$ -marked basis, even when  $(G)$  and  $J$  share the same Hilbert function. Moreover, it can happen that a  $J$ -marked set  $G$  is not a  $J$ -marked basis, although there exists an ideal  $I$  containing  $G$  but not generated by  $G$  such that  $\mathcal{N}(J)$  is a  $K$ -basis for  $S/I$ .

*Example 1.10.* (i) In  $K[x, y, z]$  let  $J = (xy, z^2)$  and  $I$  be the ideal generated by  $f_1 = xy + yz$ ,  $f_2 = z^2 + xz$ , which form a  $J$ -marked set. Note that  $J$  defines a 0-dimensional subscheme in  $\mathbb{P}^2$ , while  $I$  defines a 1-dimensional subscheme, because it contains the line  $x + z = 0$ . Therefore,  $I$  and  $J$  do not have the same Hilbert function, so that  $\{f_1, f_2\}$  is not a  $J$ -marked basis by Remark 1.5.

(ii) In  $K[x, y, z]$ , let  $J = (xy, z^2)$  and  $I$  be the ideal generated by  $g_1 = xy + x^2 - yz$ ,  $g_2 = z^2 + y^2 - xz$ , which form a  $J$ -marked set. Note that  $J$  and  $I$  have the same Hilbert function because they are both complete intersections of two quadrics. However,  $\mathcal{N}(J)$  is not free in  $K[x, y, z]/I$  because  $zg_1 + yg_2 = x^2z + y^3 \in I$  is a sum of terms in  $\mathcal{N}(J)$ . Hence  $\{g_1, g_2\}$  is not a  $J$ -marked basis.

(iii) In  $K[x, y, z]$ , let  $J = (xy, z^2)$  and  $I$  be the ideal generated by  $f_1 = xy + yz, f_2 = z^2 + xz, f_3 = xyz$ . Both  $I$  and  $J$  define 0-dimensional subschemes in  $\mathbb{P}^2$  of degree 4. Moreover,  $I$  belongs to  $\mathcal{Mf}(J)$  because for every  $m \geq 2$  the  $K$ -vector space  $U_m = I_m + \mathcal{N}(J)_m = I_m + \langle x^m, y^m, x^{m-1}z, y^{m-1}z \rangle$  is equal to  $K[x, y, z]_m$ . This is obvious for  $m = 2$ . Assume  $m \geq 3$ . Then,  $U_m$  contains all the terms  $y^{m-i}z^i$ , because  $yz^2 = zf_1 - f_3$  belongs to  $I$ . Moreover  $U_m$  contains all the terms  $x^{m-i}y^i$  because  $x^2y = xf_1 - f_3 \in I$  and  $xy^{m-1} = y^{m-2}f_1 - zy^{m-1} \in U_m$ . Finally, by induction on  $i$ , we can see that all the terms  $x^iz^{m-i}$  belong to  $U_m$ . Indeed, as already proved,  $z^m$  belongs to  $U_m$ , hence  $x^{i-1}z^{m-i+1} \in U_m$  implies  $x^iz^{m-i} = x^{i-1}z^{m-i-1}f_2 - x^{i-1}z^{m-i+1} \in U_m$ . However, the  $J$ -marked set  $G = \{f_1, f_2\}$  does not generate  $I$  and is not a  $J$ -marked basis, as shown in (i).

## 2. STRONGLY STABLE IDEALS $J$ AND $J$ -MARKED BASES

In this section we show that the properties of  $J$ -marked sets improve decisively if  $J$  is strongly stable.

Recall that a monomial ideal  $J$  is strongly stable if and only if, for every  $x_0^{\alpha_0} \dots x_n^{\alpha_n}$  in  $J$ , also the term  $x_0^{\alpha_0} \dots x_i^{\alpha_i-1} \dots x_j^{\alpha_j+1} \dots x_n^{\alpha_n}$  belongs to  $J$ , for each  $0 \leq i < j \leq n$  with  $\alpha_i > 0$ , or, equivalently, for every  $x_0^{\beta_0} \dots x_n^{\beta_n}$  in  $\mathcal{N}(J)$ , also the term  $x_0^{\beta_0} \dots x_h^{\beta_h+1} \dots x_k^{\beta_k-1} \dots x_n^{\beta_n}$  belongs to  $\mathcal{N}(J)$ , for each  $0 \leq h < k \leq n$  with  $\beta_k > 0$ .

A strongly stable ideal is always Borel-fixed, that is fixed under the action of the Borel subgroup of lower-triangular invertibles matrices. If  $ch(K) = 0$ , also the vice versa holds (e.g. [5]) and [7] guarantees that in generic coordinates the initial ideal of an ideal  $I$ , with respect to a fixed term order, is a constant Borel-fixed monomial ideal, denoted  $gin(I)$  and called the *generic initial ideal* of  $I$ .

In [17] a *reduction relation*  $\xrightarrow{\mathcal{F}}$  modulo a given set  $\mathcal{F}$  of marked polynomials is defined in the usual sense of Gröbner bases theory and it is proved that, if  $\xrightarrow{\mathcal{F}}$  is Noetherian, then there exists an admissible term order  $\prec$  on  $S$  such that  $Ht(f)$  is the  $\prec$ -leading term of  $f$ , for all  $f \in \mathcal{F}$ , being the converse already known [3]. If we take a  $J$ -marked set  $G$ ,  $\xrightarrow{G}$  can be non-Noetherian, as the following example shows. However, we will see that, if  $J$  is a strongly stable ideal and  $G$  is a  $J$ -marked set, every homogeneous polynomial has a  $J$ -normal form modulo  $(G)$ .

*Example 2.1.* Let us consider the  $J$ -marked set  $G = \{f_1 = xy + yz, f_2 = z^2 + xz\}$ , where  $Ht(f_1) = xy$  and  $Ht(f_2) = z^2$ . The term  $h = xyz$  can be rewritten only by  $xyz - zf_1 = -yz^2$  and the term  $-yz^2$  can be rewritten only by  $-yz^2 + yf_2 = xyz$ , which is again the term we wanted to rewrite. Hence, the reduction relation  $\xrightarrow{G}$  is not Noetherian. Observe that in this case  $J = (xy, z^2)$  is not strongly stable, but  $\xrightarrow{G}$  can be non-Noetherian also if  $J$  is strongly stable, as Example 3.18 will show.

**Theorem 2.2.** (Existence of  $J$ -normal forms) *Let  $G = \{f_\alpha = x^\alpha - \sum c_{\alpha\gamma}x^\gamma : Ht(f_\alpha) = x^\alpha \in B_J\}$  be a  $J$ -marked set, with  $J$  strongly stable. Then, every polynomial of  $S$  has a  $J$ -normal form modulo  $(G)$ .*

*Proof.* It is sufficient to prove that our assertion holds for the terms, because  $G$  is formed by homogeneous polynomials. Let us consider the set  $E$  of terms which have not a  $J$ -normal form modulo  $(G)$ . Of course  $E \cap B_J = \emptyset$ . If  $E$  is not empty and  $x^\beta$  belongs to  $E$ , then  $x^\beta = x_i x^\delta$  for some  $x^\delta$  in  $J$ . We choose  $x^\beta$  so that its degree  $m$  is the minimum in  $E$  and that, among the

terms of degree  $m$  in  $E$ ,  $x_i$  is minimal. Let  $\sum c_{\delta\gamma}x^\gamma$  be a  $J$ -normal form modulo  $(G)$  of  $x^\delta$ , that exists by the minimality of  $m$ . Thus we can rewrite  $x^\beta$  by  $\sum c_{\delta\gamma}x_i x^\gamma$ . We claim that all terms  $x_i x^\gamma$  do not belong to  $E$ . On the contrary, if  $x_i x^\gamma$  belongs to  $E$ , then  $x_i x^\gamma = x_j x^\epsilon$  for some  $x^\epsilon$  in  $J$ . If it were  $x_i < x_j$  then, by the strongly stable property and since  $x^\gamma$  belongs to  $\mathcal{N}(J)$ , we would get that  $x^\epsilon = x_i x^\gamma / x_j$  belongs to  $\mathcal{N}(J)$ , that is impossible. So, we have  $x_j < x_i$  and by the minimality of  $x_i$  the term  $x_i x^\gamma$  has a  $J$ -normal form modulo  $(G)$ . This is a contradiction and so  $E$  is empty.  $\square$

**Corollary 2.3.** *If  $J$  is a strongly stable ideal and  $I$  a homogeneous ideal containing a  $J$ -marked set  $G$ , then  $\mathcal{N}(J)$  generates  $S/I$  as a  $K$ -vector space, so that  $\dim_K I_m \geq \dim_K J_m$ , for every  $m \geq 0$ .*

*Proof.* By Theorem 2.2, for every polynomial  $h$  there exists a polynomial  $h_0$  such that  $h - h_0$  belongs to  $(G) \subseteq I$  and  $\text{Supp}(h_0) \subseteq \mathcal{N}(J)$ . So, all the elements of  $S/I$  are linear combinations of terms of  $\mathcal{N}(J)$  and the thesis follows.  $\square$

**Corollary 2.4.** *Let  $J$  be a strongly stable ideal and  $G$  be a  $J$ -marked set. Then,  $G$  is a  $J$ -marked basis if and only if  $\dim_K(G)_m \leq \dim_K J_m$ , for every  $m \geq 0$  or, equivalently,  $\mathcal{N}(J)$  is free in  $S/(G)$ .*

*Proof.* By Proposition 1.8,  $G$  is a  $J$ -marked basis if and only if every polynomial has a unique  $J$ -normal form modulo  $(G)$ . So, it is enough to apply Theorem 2.2 and Corollary 2.3.  $\square$

**Corollary 2.5.** *Let  $J$  be a strongly stable ideal and  $I$  be a homogeneous ideal. Then  $I$  belongs to  $\mathcal{Mf}(J)$  if and only if  $I$  has a  $J$ -marked basis.*

*Proof.* If  $I$  has a  $J$ -marked basis then  $I$  belongs to  $\mathcal{Mf}(J)$  by definition. Vice versa, apply Remark 1.7(1) and Corollary 2.4.  $\square$

*Remark 2.6.* Every reduced Gröbner basis of a homogeneous ideal with respect to a graded term order is a  $J$ -marked basis for some monomial ideal  $J$ , hence every homogeneous ideal contains a  $J$ -marked basis. But, unless we are in generic coordinates, not every (homogeneous) ideal contains a  $J$ -marked basis with  $J$  strongly stable, as for example a monomial ideal which is not strongly stable.

Let  $G$  be a  $J$ -marked basis with  $J$  strongly stable. Thanks to the existence and the unicity of  $J$ -normal forms,  $G$  can behave like a Gröbner basis in solving problems, as the membership ideal problem in the homogeneous case. Indeed, by the unicity of  $J$ -normal forms, a polynomial belongs to the ideal  $(G)$  if and only if its  $J$ -normal form modulo  $(G)$  is null. But, until now, we have not a computational method to construct  $J$ -normal forms yet.

In next section, by exploiting the proof of Theorem 2.2, we provide an algorithm which, in the hypothesis that  $J$  is strongly stable, reduces every homogeneous polynomial to a  $J$ -normal form modulo  $(G)$  in a finite number of steps, although  $\xrightarrow{G}$  is not necessarily Noetherian. This fact allows us also to recognize when a  $J$ -marked set is a  $J$ -marked basis by a Buchberger-like criterion and, hence, to develop effective computational aspects of  $J$ -marked bases.

### 3. EFFECTIVE METHODS FOR $J$ -MARKED BASES

Let  $I$  be the homogeneous ideal generated by a  $J$ -marked set  $G = \{f_\alpha = x^\alpha - \sum c_{\alpha\gamma}x^\gamma : Ht(f_\alpha) = x^\alpha \in B_J\}$ , where  $J$  is strongly stable, so that every polynomial has a  $J$ -normal form modulo  $I$ , by Theorem 2.2.

In this section we obtain an efficient procedure to compute in a finite number of steps a  $J$ -normal form modulo  $I$  of every homogeneous polynomial. To this aim, we need some more definitions and results.

For every degree  $m$ , the  $K$ -vector space  $I_m$  formed by the homogeneous polynomials of degree  $m$  of  $I$  is generated by the set  $W_m = \{x^\delta f_\alpha : x^{\delta+\alpha} \text{ has degree } m, f_\alpha \in G\}$ , that becomes a set of marked polynomials letting  $Ht(x^\delta f_\alpha) = x^{\delta+\alpha}$ .

**Lemma 3.1.** *Let  $x^\beta$  be a term of  $J_m \setminus B_J$  and  $x_i = \min(x^\beta)$ . Then  $x^\beta/x_i$  belongs to  $J_{m-1}$ .*

*Proof.* By the hypothesis there exists at least a term of  $J_{m-1}$  that divides the given term  $x^\beta$ . So, let  $x_j$  such that  $x^\beta/x_j$  belongs to  $J_{m-1}$ . If  $x_j = x_i$ , we are done. Otherwise, we get  $x^\beta = x_i x_j x^\delta$ , for some term  $x^\delta$ , so that  $x_i x^\delta = x^\beta/x_j$  belongs to  $J_{m-1}$ . By the definition of a strongly stable ideal and since  $x_j > x_i$ , we obtain that  $x^\beta/x_i = x_j x^\delta$  belongs to  $J_{m-1}$ .  $\square$

**Definition 3.2.** For every  $m \geq \alpha_J$ , we define the following special subset of  $W_m$ , by induction on  $m$ . If  $m = \alpha_J$  is the initial degree of  $J$ , we set  $V_m := G_m$ . For every  $m > \alpha_J$ , we set  $V_m := G_m \cup \{g_\beta : x^\beta \in J_m \setminus G_m\}$ , where  $g_\beta := x_i g_\epsilon$  with  $x_i = \min(x^\beta)$  and  $g_\epsilon$  the unique polynomial of  $V_{m-1}$  with head term  $x^\epsilon = x^\beta/x_i$ .

*Remark 3.3.* By construction, for every element  $g_\beta$  of  $V_m \subseteq W_m$  there exist  $x^\delta$  and  $f_\alpha \in G$  such that  $g_\beta = x^\delta f_\alpha$  and  $x^\delta = 1$  or  $\max(x^\delta) \leq \min(x^\alpha)$ . In particular, we get  $\min(x^\delta) = \min(x^\beta)$ . Note that Definition 3.2 makes sense due to Lemma 3.1.

For every integer  $m \geq \alpha_J$ , we define the following total order  $\succeq_m$  on  $V_m$ .

**Definition 3.4.** For every  $f_\alpha, f_{\alpha'} \in G_m$ , we set  $f_\alpha \succeq_m f_{\alpha'}$  if and only if  $Ht(f_\alpha) \geq Ht(f_{\alpha'})$  with respect to a fixed term order  $\geq$ . For every  $g_\beta \in V_m \setminus G_m$  and  $f_\alpha \in G_m$ , we set  $g_\beta \succeq_m f_\alpha$ . For every  $m > \alpha_J$ , given  $x_i g_\epsilon, x_j g_\eta \in V_m \setminus G_m$ , where  $x_i = \min(x_i x^\epsilon)$  and  $x_j = \min(x_j x^\eta)$ , we set

$$x_i g_\epsilon \succeq_m x_j g_\eta \Leftrightarrow x_i > x_j \text{ or } x_i = x_j \text{ and } g_\epsilon \succeq_{m-1} g_\eta.$$

By the definition of  $V_m$  and by well-known properties of a strongly stable ideal, we get the routine **VCONSTRUCTOR** to compute  $V_m$ , for every  $\alpha_J \leq m \leq s$ .

**Lemma 3.5.** *With the above notation,*

$$x_i g_\epsilon \in V_m \setminus G_m \text{ and } x^\beta \in \text{Supp}(x_i g_\epsilon) \setminus \{x_i x^\epsilon\} \text{ with } g_\beta \in V_m \Rightarrow x_i g_\epsilon \succ_m g_\beta.$$

*Proof.* By induction on  $m$ , first observe that for  $m = \alpha_J$  there is nothing to prove because  $V_{\alpha_J} = G_{\alpha_J}$ . For  $m > \alpha_J$ , let  $g_\beta = x_j g_\eta \notin G_m$ . If  $x_i = x_j$ , then  $x^\eta$  belongs to  $\text{Supp}(g_\epsilon) \setminus \{x^\epsilon\}$  and, by the induction, we have  $g_\eta \prec_{m-1} g_\epsilon$ . Otherwise, note that every term of  $\text{Supp}(x_i g_\epsilon)$  is divided by  $x_i$ , so  $x_j x^\eta = x_i x^\lambda$  and, by Remark 3.3, we get  $x_j = \min(x^\beta) = \min(x_i x^\lambda) \leq x_i$ .  $\square$

**Proposition 3.6.** (Construction of  $J$ -normal forms) *With the above notation, every term  $x^\beta \in J_m \setminus G_m$  can be reduced to a  $J$ -normal form modulo  $I$  in a finite number of reduction steps, using only polynomials of  $V_m$ . Hence, the reduction relation  $\xrightarrow{V_m}$  is Noetherian in  $S_m$ .*

*Proof.* By definition of  $V_m$ , every term  $x^\beta$  of  $J_m$  is the head term of one and only one polynomial  $g_\beta$  of  $V_m \subseteq W_m$ . Hence, we rewrite  $x^\beta$  by  $g_\beta$  getting a  $K$ -linear combination of terms belonging to  $\text{Supp}(g_\beta) \setminus \{x^\beta\}$ . Applying Lemma 3.5 repeatedly, we are done since  $V_m$  is a finite set.  $\square$

---

```

1: procedure VCONSTRUCTOR( $G, s$ )  $\rightarrow V_{\alpha_j} \dots, V_s$ 
Require:  $G$  is a  $J$ -marked set ordered with respect to a graduate term order on the head
        terms, with  $J$  a strongly stable ideal, and  $s \geq \alpha_J$ .
Ensure:  $V_m$  ordered by  $\succeq_m$ , for every  $\alpha_J \leq m \leq s$ 
2:    $\alpha_J := \min\{\deg(Ht(f_\alpha)) \mid f_\alpha \in G\}$ 
3:    $V_{\alpha_J} := G_{\alpha_J}$ 
4:   for  $m = \alpha_J + 1$  to  $s$  do
5:      $V_m := G_m$ ;
6:     for  $i = 0$  to  $n$  do
7:       for  $j = 1$  to  $|V_{m-1}|$  do
8:         if  $i \leq \min(Ht(V_{m-1}[j]))$  then
9:            $V_m = V_m \cup \{x_i V_{m-1}[j]\}$ 
10:        end if
11:      end for
12:    end for
13:  end for
14:  return  $V_{\alpha_j} \dots, V_s$ ;
15: end procedure

```

---

**Definition 3.7.** A homogeneous polynomial, with support contained in  $\mathcal{N}(J)$  and in relation by  $\xrightarrow{V_m}$  to a homogeneous polynomial  $h$  of degree  $m$ , is denoted  $\bar{h}$  and called  $V_m$ -reduction of  $h$ .

For every homogeneous polynomial  $h$  of degree  $m$ ,  $\bar{h}$  is a  $J$ -normal form modulo  $I$ . Hence, from the procedure described in the proof of Proposition 3.6 we obtain the routine NORMALFORMCONSTRUCTOR that, actually, form a step of a division algorithm with respect to a  $J$ -marked set, with  $J$  strongly stable.

---

```

1: procedure NORMALFORMCONSTRUCTOR( $h, V_m$ )  $\rightarrow \bar{h}$ 
Require:  $h$  is a homogeneous polynomial of degree  $m$ 
Require: a list  $V_m$ , as defined in Definition 3.2, and ordered by  $\succeq_m$ 
Ensure:  $V_m$ -reduction  $\bar{h}$  of  $h$ 
2:    $L := |V_m|$ ;
3:   for  $K = 1$  to  $L$  do
4:      $x^\eta := Ht(V_m[K])$ ;
5:      $a :=$ coefficient of  $x^\eta$  in  $h$ ;
6:     if  $a \neq 0$  then
7:        $h := h - a \cdot V_m[K]$ ;
8:     end if;
9:   end for
10:  return  $h$ ;
11: end procedure

```

---

*Remark 3.8.* There is a strong analogy between the union of the sets  $V_m$  and the so-called *staggered bases*, introduced by [8] and studied also by [15].

Now, we extend to  $W_m$  the order  $\succeq_m$  defined on  $V_m$ . Recall that, in our setting, a term  $x^\delta$  is higher than a term  $x^{\delta'}$  with respect to the degree reverse lexicographic term order (for short  $x^\delta >_{drl} x^{\delta'}$ ) if  $|\delta| > |\delta'|$  or  $|\delta| = |\delta'|$  and the first non null entry of  $\delta - \delta'$  is negative.

**Definition 3.9.** Let the polynomials of  $G_m$  be ordered as in Definition 3.4 and  $x^\delta f_\alpha, x^{\delta'} f_{\alpha'}$  be two elements of  $W_m$ . We set

$$x^\delta f_\alpha \succeq_m x^{\delta'} f_{\alpha'} \Leftrightarrow x^\delta >_{drl} x^{\delta'} \text{ or } x^\delta = x^{\delta'} \text{ and } Ht(f_\alpha) \geq Ht(f_{\alpha'}).$$

**Lemma 3.10.** (i) For every two elements  $x^\delta f_\alpha, x^{\delta'} f_{\alpha'}$  of  $W_m$  we get

$$x^\delta f_\alpha \succeq_m x^{\delta'} f_{\alpha'} \Rightarrow \forall x^\eta : x^{\delta+\eta} f_\alpha \succeq_{m'} x^{\delta'+\eta} f_{\alpha'},$$

where  $m' = |\delta + \eta + \alpha|$ .

(ii) Every polynomial  $g_\beta \in V_m$  is the minimum with respect to  $\preceq_m$  of the subset  $W_\beta$  of  $W_m$  containing all polynomials of  $W_m$  with  $x^\beta$  as head term.

(iii)  $x^\delta f_\alpha \in W_m \setminus G_m$  and  $x^\beta \in \text{Supp}(x^\delta f_\alpha) \setminus \{x^\delta f_\alpha\}$  with  $g_\beta \in V_m \Rightarrow x^\delta f_\alpha \succ_m g_\beta$ .

*Proof.* (i) It follows by the analogous property of every term order.

(ii) The statement holds by construction of  $V_m$  and by Remark 3.3. Indeed, by same arguments as before, if  $x^\delta f_\alpha$  is any polynomial of  $W_\beta$  and  $g_\beta = x^{\delta'} f_{\alpha'} \in V_m$ , with  $\max(x^{\delta'}) \leq \min(x^{\alpha'})$  as in Remark 3.3, then  $x_j = \min(x^{\delta'}) = \min(x^{\delta'+\alpha'}) = \min(x^{\delta+\alpha}) \leq \min(x^\delta)$ . If the equality holds, it is enough to observe that  $\frac{x^\delta}{x_j} f_\alpha \in W_{m-1}$  and  $\frac{x^{\delta'}}{x_j} f_{\alpha'} \in V_{m-1}$  by construction.

(iii) It is analogous to the proof of Lemma 3.5. If  $x^\beta$  belongs to  $J_m$  we are done. Otherwise, note that every term of  $\text{Supp}(x^\delta f_\alpha)$  is multiple of  $x^\delta$ , in particular  $x^{\delta'+\alpha'} = x^{\delta+\gamma}$  for some  $x^\gamma \in \mathcal{N}(J)$ . Let  $x_i = \min(x^\delta)$  and  $x_j = \min(x^{\delta'})$ . By Remark 3.3, we get  $x_j = \min(x^{\delta'+\alpha'}) = \min(x^{\delta+\gamma}) \leq \min(x^\delta) = x_i$ . If  $x_j = x_i$ , then  $x^\beta/x_i$  belongs to the support of  $\frac{x^\delta}{x_i} f_\alpha$  and use induction.  $\square$

In Remark 2.6 we have already observed that in generic coordinates every homogeneous ideal has a  $J$ -marked basis, with  $J$  strongly stable. Now, given a strongly stable ideal  $J$ , we describe a *Buchberger-like* algorithmic method to check if a  $J$ -marked set is or not a  $J$ -marked basis, recovering the well-known notion of  $S$ -polynomial from the Gröbner bases theory.

**Definition 3.11.** The  $S$ -polynomial of two elements  $f_\alpha, f_{\alpha'}$  of a  $J$ -marked set  $G$  is the polynomial  $S(f_\alpha, f_{\alpha'}) := x^\beta f_\alpha - x^{\beta'} f_{\alpha'}$ , where  $x^{\beta+\alpha} = x^{\beta'+\alpha'} = \text{lcm}(x^\alpha, x^{\alpha'})$ .

**Theorem 3.12.** (Buchberger-like criterion) Let  $J$  be a strongly stable ideal and  $I$  the homogeneous ideal generated by a  $J$ -marked set  $G$ . With the above notation:

$$I \in \mathcal{Mf}(J) \Leftrightarrow \overline{S(f_\alpha, f_{\alpha'})} = 0, \forall f_\alpha, f_{\alpha'} \in G.$$

*Proof.* Recall that  $I \in \mathcal{Mf}(J)$  if and only if  $G$  is a  $J$ -marked basis, so that every polynomial has a unique  $J$ -normal form modulo  $I$ . Since  $\overline{S(f_\alpha, f_{\alpha'})}$  belongs to  $I$  by construction, its  $J$ -normal form modulo  $I$  is null and coincides with  $\overline{S(f_\alpha, f_{\alpha'})}$ , by the unicity of  $J$ -normal forms.

For the converse, by Corollary 2.4 it is enough to show that, for every  $m$ , the  $K$ -vector space  $I_m$  is generated by the  $\dim_K J_m$  elements of  $V_m$ . More precisely we will show that every polynomial  $x^\delta f_\alpha \in W_m$  either belongs to  $V_m$  or is a  $K$ -linear combination of elements of  $V_m$  lower than  $x^\delta f_\alpha$  itself. We may assume that this fact holds for every polynomial in  $W_m$  lower than  $x^\delta f_\alpha$ . If  $x^\delta f_\alpha$  belongs to  $V_m$  there is nothing to prove. If  $x^\delta f_\alpha$  does not belong

to  $V_m$ , let  $x^{\delta'} f_{\alpha'} = \min(W_{\delta+\alpha}) \in V_m$ , so that  $x^{\delta} f_{\alpha} \succ_m x^{\delta'} f_{\alpha'}$ , and consider the polynomial  $g = x^{\delta} f_{\alpha} - x^{\delta'} f_{\alpha'}$ .

If  $g$  is the  $S$ -polynomial  $S(f_{\alpha}, f_{\alpha'})$ , then it is a  $K$ -linear combination  $\sum c_i g_{\eta_i}$  of polynomials of  $V_m$  because  $\overline{S(f_{\alpha}, f_{\alpha'})} = 0$  by the hypothesis. Moreover, by construction,  $x^{\delta'} f_{\alpha'}$  belongs to  $V_m$  and, thanks to Lemma 3.10, for all  $i$  we have  $x^{\delta} f_{\alpha} \succ_m g_{\eta_i}$ .

If  $g$  is not the  $S$ -polynomial  $S(f_{\alpha}, f_{\alpha'})$ , then there exists a term  $x^{\beta} \neq 1$  such that  $g = x^{\beta} S(f_{\alpha}, f_{\alpha'}) = x^{\beta} (x^{\eta} f_{\alpha} - x^{\eta'} f_{\alpha'})$ , where recall that by the hypothesis  $S(f_{\alpha}, f_{\alpha'})$  is a  $K$ -linear combination  $\sum c_i g_{\eta_i}$  of elements of  $V_{m-|\beta|}$  lower than  $x^{\eta} f_{\alpha}$ , being again  $x^{\eta} f_{\alpha} \succ_m x^{\eta'} f_{\alpha'}$ . Hence,  $x^{\delta} f_{\alpha} = x^{\delta'} f_{\alpha'} + \sum c_i x^{\beta} g_{\eta_i}$ , where all polynomials appearing in the right hand are lower than  $x^{\delta} f_{\alpha}$  with respect to  $\succ_m$ . So we can apply to them the inductive hypothesis for which either they are elements of  $V_m$  or they are  $K$ -linear combinations of lower elements in  $V_m$ . This allows us to conclude.  $\square$

Let  $H = (h_1, \dots, h_t)$  be a syzygy of a  $J$ -basis  $G = \{f_{\alpha_1}, \dots, f_{\alpha_t}\}$  such that every polynomial  $h_i = \sum c_{i\beta} x^{\beta}$  is homogeneous and every product  $h_i f_{\alpha_i}$  has the same degree  $m$ . A syzygy  $M = (m_1, \dots, m_t)$  of  $J$  is homogeneous if, for every  $1 \leq i \leq t$ , we have  $m_i x^{\alpha_i} = c_{i\epsilon} x^{\epsilon}$ , for a constant term  $x^{\epsilon}$  and  $c_{i\epsilon} \in K$ .

**Definition 3.13.** The *head term*  $Ht(H)$  of the syzygy  $H$  is the head term of the polynomial  $H_{\max} := \max_{\preceq_m} \{x^{\beta} f_{\alpha_i} : i \in \{1, \dots, t\}, x^{\beta} \in \text{Supp}(h_i)\}$ . If  $Ht(H) = x^{\eta}$ , let  $H^+ = (h_1^+, \dots, h_t^+)$  be the  $t$ -uple such that  $h_i^+ = c_{i\beta} x^{\beta}$ , where  $x^{\beta} x^{\alpha_i} = x^{\eta}$ , i.e.  $x^{\beta} f_{\alpha_i} \in W_{\eta}$ . Given a homogeneous syzygy  $M$  of  $J$ , we say that  $H$  is a *lifting* of  $M$ , or that  $M$  *lifts to*  $H$ , if  $H^+ = M$ .

**Corollary 3.14.** *Every homogeneous syzygy of  $J$  lifts to a syzygy of a  $J$ -marked basis  $G$ .*

*Proof.* Recall that syzygies of type  $(0, \dots, x^{\beta}, \dots, -x^{\beta'}, 0, \dots)$  form a system of homogeneous generators of syzygies of  $B_J = \{\dots, x^{\alpha}, \dots, x^{\alpha'}, \dots\}$ , where  $x^{\beta+\alpha} = x^{\beta'+\alpha'} = \text{lcm}(x^{\alpha}, x^{\alpha'})$ . Thus, apply Theorem 3.12.  $\square$

Until now we have shown that a  $J$ -marked basis satisfies the characterizing properties of a Gröbner basis. In the following result we consider a property that does not characterize Gröbner bases, but it is satisfied by Gröbner bases. We show that it is satisfied by  $J$ -marked bases too, by standard arguments.

**Corollary 3.15.** *Let  $\{M_1, \dots, M_t\}$  be a set of homogeneous generators of the module of syzygies of  $J$ . Then, a set  $\{K_1, \dots, K_t\}$  of liftings of the  $M_i$ 's generates the module of syzygies of  $G$ .*

*Proof.* First, observe that the module of syzygies of  $G = \{f_{\alpha_1}, \dots, f_{\alpha_t}\}$  is generated by the syzygies  $H = (h_1, \dots, h_t)$  such that every  $h_i = \sum c_{i\beta} x^{\beta}$  is a homogeneous polynomial and every product  $h_i f_{\alpha_i}$  has the same degree  $m$ . Let  $H^+$  the syzygy of  $J$ , as computed in Definition 3.13. Hence, there exist homogeneous polynomials  $q_1, \dots, q_t$  such that  $H^+ = \sum q_i M_i$ . Let  $H_1 = H - \sum q_i K_i$ . By construction we get that  $H_{\max}(H_1) \prec_m H_{\max}(H)$ , by Lemmas 3.5 and 3.10. Since  $\preceq_m$  is a total order on the finite set  $W_m$ , we can conclude.  $\square$

**Remark 3.16.** In the proof of Theorem 3.12 we do not use  $V_m$ -reductions of all  $S$ -polynomials  $x^{\delta} f_{\alpha} - x^{\delta'} f_{\alpha'}$  of elements in  $G$ , but only of those such that either  $x^{\delta} f_{\alpha}$  or  $x^{\delta'} f_{\alpha'}$  belongs to some  $V_m$ . Moreover, we can consider the analogous property to that of the improved Buchberger algorithm that only considers  $S$ -polynomials corresponding to a set of generators for the syzygies

of  $J$ . Thus we can improve Corollary 2.4 and say that, in the same hypotheses:

$$I \in \mathcal{Mf}(J) \iff \forall m \leq m_0, \dim_k I_m = \dim_k J_m \iff \forall m \leq m_0, \dim_k I_m \leq \dim_k J_m$$

where  $m_0$  is the maximum degree of generators of syzygies of  $J$ . Hence, to prove that  $\dim_k I_m = \dim_k J_m$  for some  $m$  it is sufficient that the  $V_m$ -reductions of the  $S$ -polynomials of degree  $\leq m$  are null.

*Example 3.17.* Let  $J = (x^2, xy, xz, y^2) \subset k[x, y, z]$ , where  $x > y > z$  and consider a  $J$ -marked set  $G = \{f_{x^2}, f_{xy}, f_{xz}, f_{y^2}\}$ . In order to check whether  $G$  is a  $J$ -marked basis it is sufficient to verify if the polynomials  $S(f_{x^2}, f_{xy})$ ,  $S(f_{x^2}, f_{xz})$ ,  $S(f_{x^2}, f_{y^2})$ ,  $S(f_{xy}, f_{xz})$  and  $S(f_{xy}, f_{y^2})$  have  $V_m$ -reductions null, but it is not necessary to controll  $S(f_{xz}, f_{y^2})$  because  $yzf_{xy}$  is the element of  $V_3$  with head term  $xy^2z$ .

*Example 3.18.* Let  $J = (x^3, x^2y, xy^2, y^5)_{\geq 4}$  be a strongly stable ideal in  $k[x, y, z]$ , with  $x > y > z$ , and  $G = B_J \cup \{f\} \setminus \{xy^2z\}$  a  $J$ -marked set, where  $f = xy^2z - y^4 - x^2z^2$  with  $Ht(f) = xy^2z$ . We can verify that  $G$  is a  $J$ -marked basis using the Buchberger-like criterion proved in Theorem 3.12. Indeed, the  $S$ -polynomials non involving  $f$  vanish and all the  $S$ -polynomials involving  $f$  are multiple of either  $x \cdot (y^4 + x^2z^2)$  or  $y \cdot (y^4 + x^2z^2)$ . Since the terms  $y^4 \cdot x$ ,  $y^4 \cdot y$ ,  $x^2z^2 \cdot x$ ,  $x^2z^2 \cdot y$  belong to  $V_5$ , all the  $S$ -polynomials have  $V_m$ -reductions null. Notice also that, in this case,  $\xrightarrow{G}$  is not Noetherian because, although the  $V_7$ -reduction of  $x^2y^2z^3$  is 0, being  $z^2 \cdot x^2y^2z \in V_7$  (while  $xzf \notin V_7$ ), a different choice of reduction gives the loop:

$$x^2y^2z^3 \xrightarrow{f} xy^4z^2 + x^3z^4 \xrightarrow{x^3z^2} xy^4z^2 \xrightarrow{f} y^6z + x^2y^2z^3 \xrightarrow{y^5} x^2y^2z^3.$$

Moreover,  $G$  is not a Gröbner basis with respect to any term order  $\prec$ . Indeed,  $xy^2z^2 \succ y^4z$  and  $xy^2z^2 \succ x^2z^3$  would be in contradiction with the equality  $(xy^2z^2)^2 = x^2z^3 \cdot y^4z$ .

#### 4. $J$ -MARKED FAMILIES AS AFFINE SCHEMES

In this section  $J$  is always supposed strongly stable, so that we can use all results described in the previous sections for  $J$ -marked bases.

Here we provide the construction of an affine scheme whose points correspond, one to one, to the ideals of the  $J$ -marked family  $\mathcal{Mf}(J)$ . Recall that  $\mathcal{Mf}(J)$  is the family of all homogeneous ideals  $I$  such that  $\mathcal{N}(J)$  is a basis for  $S/I$  as a  $K$ -vector space, hence  $\mathcal{Mf}(J)$  contains all homogeneous ideals for which  $J$  is the initial ideal with respect to a fixed term order. We generalize to any strongly stable ideal  $J$  an approach already proposed in literature in case  $J$  is considered an initial ideal (e.g. [4, 6, 11, 18, 19]).

For every  $x^\alpha \in B_J$ , let  $F_\alpha := x^\alpha - \sum C_{\alpha\gamma}x^\gamma$ , where  $x^\gamma$  belongs to  $\mathcal{N}(J)_{|\alpha|}$  and the  $C_{\alpha\gamma}$ 's are new variables. Let  $C$  be the set of such new variables and  $N := |C|$ . The set  $\mathcal{G}$  of all the polynomials  $F_\alpha$  becomes a  $J$ -marked set letting  $Ht(F_\alpha) = x^\alpha$ . From  $\mathcal{G}$  we can obtain the  $J$ -marked basis of every ideal  $I \in \mathcal{Mf}(J)$  specializing in a unique way the variables  $C$  in  $K^N$ , since every ideal  $I \in \mathcal{Mf}(J)$  has a unique  $J$ -marked basis (Remark 1.9 and Corollary 2.5). But not every specialization gives rise to an ideal of  $\mathcal{Mf}(J)$ .

Let  $\mathcal{V}_m$  be the analogous for  $\mathcal{G}$  of  $V_m$  for any  $G$ . Let  $H_{\alpha\alpha'}$  be the  $\mathcal{V}_m$ -reductions of the  $S$ -polynomials  $S(F_\alpha, F_{\alpha'})$  of elements of  $\mathcal{G}$  and extract their coefficients that are polynomials in  $K[C]$ . We will denote by  $\mathfrak{R}$  the ideal of  $K[C]$  generated by these coefficients. Let  $\mathfrak{R}'$  be the ideal of  $K[C]$  obtained in the same way of  $\mathfrak{R}$  but only considering  $S$ -polynomials  $S(F_\alpha, F_{\alpha'}) = x^\delta F_\alpha - x^{\delta'} F_{\alpha'}$  such that  $x^\delta F_\alpha$  is minimal among those with head term  $x^{\delta+\alpha}$ .

**Theorem 4.1.** *There is a one to one correspondence between the ideals of  $\mathcal{Mf}(J)$  and the points of the affine scheme in  $K^N$  defined by the ideal  $\mathfrak{R}$ . Moreover,  $\mathfrak{R}' = \mathfrak{R}$ .*

*Proof.* For the first assertion it is enough to apply Theorem 3.12, observing that a specialization of the variables  $C$  in  $K^N$  gives rise to a  $J$ -marked basis if and only if the values chosen for the variables  $C$  form a point of  $K^N$  on which all polynomials of the ideal  $\mathfrak{R}$  vanish.

For the second assertion, first recall that, by Remark 3.16, every  $S$ -polynomial  $x^\delta F_\alpha - x^{\delta'} F_{\alpha'}$  can be written as the sum  $(x^\delta F_\alpha - x^{\delta''} F_{\alpha''}) + (x^{\delta''} F_{\alpha''} - x^{\delta'} F_{\alpha'})$  of two  $S$ -polynomials, where  $x^{\delta''} F_{\alpha''}$  belongs to  $V_m$ . Note that, considering the variables  $C$  as parameters, the support of  $x^\delta F_\alpha - x^{\delta'} F_{\alpha'}$  is contained in the union of the supports of  $x^\delta F_\alpha - x^{\delta''} F_{\alpha''}$  and of  $x^{\delta''} F_{\alpha''} - x^{\delta'} F_{\alpha'}$ . In particular, the coefficients in  $x^\delta F_\alpha - x^{\delta'} F_{\alpha'}$ , i.e. the generators of  $\mathfrak{R}$ , are combinations of the coefficients in  $(x^\delta F_\alpha - x^{\delta''} F_{\alpha''}) + (x^{\delta''} F_{\alpha''} - x^{\delta'} F_{\alpha'})$ , i.e. of the generators of  $\mathfrak{R}'$ .  $\square$

Now, by exploiting ideas of [11], we show how to obtain  $\mathfrak{R}$  in a different way, using the rank of some matrices.

By Corollary 2.4, a specialization  $C \rightarrow c \in K^N$  transforms  $\mathcal{G}$  in a  $J$ -basis  $G$  if and only if  $\dim_K(G)_m = \dim_K J_m$ , for every degree  $m$ . Thus, for each  $m$ , consider the matrix  $A_m$  whose columns correspond to the terms of degree  $m$  in  $S = K[x_1, \dots, x_n]$  and whose rows contain the coefficients of the terms in every polynomial of degree  $m$  of type  $x^\delta F_\alpha$ . Hence, every entry of the matrix  $A_m$  is 1, 0 or one of the variables  $C$ . Let  $\mathfrak{A}$  be the ideal of  $K[C]$  generated by the minors of order  $\dim_K J_m + 1$  of  $A_m$ , for every  $m$ .

**Lemma 4.2.** *The ideal  $\mathfrak{A}$  is equal to the ideal  $\mathfrak{R}'$ .*

*Proof.* Let  $a_m = \dim_K J_m$ . We consider in  $A_m$  the  $a_m \times a_m$  submatrix whose columns corresponds to the terms in  $J_m$  and whose rows are given by the polynomials  $x^\beta F_\alpha$  that are minimal with respect to the partial order  $>_m$ . Up to a permutation of rows and columns, this submatrix is upper-triangular with 1 on the main diagonal. We may also assume that it corresponds to the first  $a_m$  rows and columns in  $A_m$ . Then the ideal  $\mathfrak{A}$  is generated by the determinants of  $a_m + 1 \times a_m + 1$  sub-matrices containing that above considered. Moreover the Gaussian row-reduction of  $A_m$  with respect to the first  $a_m$  rows is nothing else than the  $\mathcal{V}_m$ -reduction of the  $S$ -polynomials of the special type considered defining  $\mathfrak{R}'$ .  $\square$

**Definition 4.3.** The affine scheme  $\mathcal{S}(J)$  defined by the ideal  $\mathfrak{R} = \mathfrak{R}' = \mathfrak{A}$  is called  *$J$ -marked scheme*.

**Theorem 4.4.** *The  $J$ -marked scheme  $\mathcal{S}(J)$  is homogeneous with respect to a non-standard grading  $\lambda$  of  $K[C]$  over the group  $\mathbb{Z}^{n+1}$  given by  $\lambda(C_{\alpha\gamma}) = \alpha - \gamma$ .*

*Proof.* To prove that  $\mathcal{Mf}(J)$  is  $\lambda$ -homogeneous it is sufficient to show that every minor of  $A_m$  is  $\lambda$ -homogeneous. Let us denote by  $C_{\alpha\alpha}$  the coefficient ( $= 1$ ) of  $x^\alpha$  in every polynomial  $F_\alpha$ : we can apply also to the “symbol”  $C_{\alpha\alpha}$  the definition of  $\lambda$ -degree of the variables  $C_{\alpha\gamma}$ , because  $\alpha - \alpha = 0$  is indeed the  $\lambda$ -degree of the constant 1. In this way, the entry in the row  $x^\beta F_\alpha$  and in the column  $x^\delta$  is  $\pm C_{\alpha\gamma}$  if  $x^\delta = x^\beta x^\gamma$  and is 0 otherwise.

Let us consider the minor of order  $s$  determined in the matrix  $A_m$  by the  $s$  rows corresponding to  $x^{\beta_i} F_{\alpha_i}$  and by the  $s$  columns corresponding to  $x^{\delta_{j_i}}$ ,  $i = 1, \dots, s$ . Every monomial that appears in the computation of such a minor is of type  $\prod_{i=1}^s C_{\alpha_i \gamma_{j_i}}$  with  $x^{\delta_{j_i}} = x^{\beta_i} x^{\gamma_{j_i}}$ . Then its

degree is:

$$\sum_{i=1}^s (\alpha_i - \gamma_{j_i}) = \sum_{i=1}^s (\alpha_i - \delta_{j_i} + \beta_i) = \sum_{i=1}^s (\alpha_i + \beta_i) - \sum_{i=1}^s \delta_{j_i}$$

which only depends on the minor.  $\square$

Let  $\prec$  be a term order and  $\mathcal{S}t_h(J, \prec)$  a so-called Gröbner stratum [11], i.e. the affine scheme that parameterizes all the homogeneous ideals with initial ideal  $J$  with respect to  $\prec$ . We can obtain  $\mathcal{S}t_h(J, \prec)$  as the section of  $\mathcal{S}(J)$  by the linear subspace  $L$  determined by the ideal  $(C_{\alpha\gamma} : x^\alpha \prec x^\gamma) \subset k[C]$ . In particular, if  $m_0$  is defined as in Remark 3.16 and, for every  $m \leq m_0$ ,  $J_m$  is a  $\prec$ -segment, i.e. it is generated by the highest  $\dim_k J_m$  monomials with respect to  $\prec$ , then  $\mathcal{S}t_h(J, \preceq)$  and  $\mathcal{S}(J)$  are the same affine scheme. In fact we can obtain both schemes using the same construction. Actually, for some strongly stable ideals  $J$  we can find a suitable term ordering such that  $\mathcal{S}t_h(J, \prec) = \mathcal{S}(J)$ , but there are cases in which  $\bigcup_{\prec} \mathcal{S}t_h(J, \prec)$  is strictly contained in  $\mathcal{S}(J)$  (see the Appendix).

The existence of a term order such that  $\mathcal{S}(J) = \mathcal{S}t_h(J, \preceq)$  has interesting consequences on the geometrical features of the affine scheme  $\mathcal{S}(J)$ . In fact the  $\lambda$ -grading on  $k[C]$  is positive if and only if such a term ordering exists and, in this case, we can isomorphically project  $\mathcal{S}(J)$  in the Zariski tangent space at the origin (see [6]). As a consequence of this projection we can prove, for instance, that the affine scheme  $\mathcal{S}(J)$  is connected and that it is isomorphic to an affine space, provided the origin is a smooth point. If for a given ideal  $J$  such a term ordering does not exist, then in general we cannot embed  $\mathcal{S}(J)$  in the Zariski tangent space at the origin (see the Appendix). However we do not know examples of Borel ideals  $J$  such that either  $\mathcal{S}(J)$  has more than one connected component or  $J$  is smooth and  $\mathcal{S}(J)$  is not rational.

Denote  $\text{reg}(I)$  the Castelnuovo-Mumford regularity of a homogeneous ideal  $I$ .

**Proposition 4.5.** *A  $J$ -marked family  $\mathcal{M}f(J)$  is flat in the origin. In particular, for every ideal  $I$  in  $\mathcal{M}f(J)$ , we get  $\text{reg}(J) \geq \text{reg}(I)$ .*

*Proof.* Analogously to what is suggested in [2] and by referring to [1, Corollary, section 3, part I], we know that  $\mathcal{M}f(J)$  is a flat family at  $J$ , i.e. at the point  $C = 0$ , if and only if every syzygy of  $J$  lifts to a syzygy among the polynomials of  $\mathcal{G}$  or, equivalently, the restrictions to  $C = 0$  of the syzygies of  $\mathcal{G}$  generate the  $S$ -module of syzygies of  $J$ . By Corollary 3.14 we know that every syzygy of  $J$  lifts to a syzygy of  $G$ , for every specialization of  $C$  in the affine scheme defined by the ideal  $\mathfrak{R}$ . And this is true thanks to Theorem 3.12 that allows also to lift a syzygy of  $J$  to a syzygy of  $\mathcal{G}$  over the ring  $(K[C]/\mathfrak{R})[x_0, \dots, x_n]$ . So, the first assertion holds.

For the second assertion, it is enough to recall that Castelnuovo-Mumford regularity is upper semicontinuous in flat families [9, Theorem 12.8, Chapter III] and that in our case the syzygies of  $J$  lift to syzygies of  $G$  for every specialization of the variables  $C$  in the affine scheme  $\mathcal{S}(J)$ , i.e. for every ideal  $I$  of  $\mathcal{M}f(J)$ , not only in some neighborhood of  $J$ .  $\square$

*Remark 4.6.* A given homogeneous ideal  $I$  belongs to  $\mathcal{M}f(J)$  if and only if  $I$  has the same Hilbert function of  $J$  and the affine scheme defined by the ideal of  $K[C]$  generated by  $\mathfrak{R}$  and by the coefficients of the  $\mathcal{V}_m$ -reductions of the generators of  $I$  is not empty. Indeed, the ideal  $I$  belongs to  $\mathcal{M}f(J)$  if and only if it has the same Hilbert function of  $J$  and there exists a specialization  $\bar{C}$  in  $\mathcal{S}(J)$  such that every generator of  $I$  belongs to the ideal  $(\bar{\mathcal{G}})$  generated by the polynomials

of  $\mathcal{G}$  evaluated on  $\bar{C}$ . The generators of  $I$  belong to  $(\bar{\mathcal{G}})$  if and only if their  $\mathcal{V}_m$ -reductions evaluated on  $\bar{C}$  become null.

#### APPENDIX: AN EXPLICIT COMPUTATION

Let  $J$  be the strongly stable ideal  $(x^4, x^3y, x^2y^2, xy^3, x^3z, x^2yz, xy^2z, y^5)$  in  $K[z, y, x]$  (where  $x > y > z$  and  $ch(K) = 0$ ), already considered in Example 3.18. Note that for every term order we can find in degree 4 a monomial in  $J$  lower than a monomial in  $\mathcal{N}(J)$ , because  $xy^2z \succ x^2z^2$  and  $xy^2z \succ y^4$  would be in contradiction with the equality  $(xy^2z)^2 = x^2z^2 \cdot y^4$ . Hence,  $J_4$  is not a segment (in the usual meaning) with respect to any term order.

The affine scheme  $\mathcal{S}(J)$  can be embedded as a locally closed subscheme in the Hilbert scheme of 8 points in the projective plane (see [12]), which is irreducible smooth of dimension 16, and contains all the Gröbner strata  $\mathcal{S}t_h(J, \prec)$ , for every  $\prec$ , and also some more point, for instance the one corresponding to the ideal  $I$  of Example 3.18.

Letting  $\mathcal{G} = \{F_1, \dots, F_8\} \subset K[x, y, z, c_1, \dots, c_{64}]$  where

$$\begin{aligned} F_1 &= x^4 + c_1z^2x^2 + c_2y^4 + c_3z^2yx + c_4zy^3 + c_5z^3x + c_6z^2y^2 + c_7z^3y + c_8z^4, \\ F_2 &= x^3y + c_9z^2x^2 + c_{10}y^4 + c_{11}z^2yx + c_{12}zy^3 + c_{13}z^3x + c_{14}z^2y^2 + c_{15}z^3y + c_{16}z^4, \\ F_3 &= x^2y^2 + c_{17}z^2x^2 + c_{18}y^4 + c_{19}z^2yx + c_{20}zy^3 + c_{21}z^3x + c_{22}z^2y^2 + c_{23}z^3y + c_{24}z^4, \\ F_4 &= xy^3 + c_{25}z^2x^2 + c_{26}y^4 + c_{27}z^2yx + c_{28}zy^3 + c_{29}z^3x + c_{30}z^2y^2 + c_{31}z^3y + c_{32}z^4, \\ F_5 &= x^3z + c_{33}z^2x^2 + c_{34}y^4 + c_{35}z^2yx + c_{36}zy^3 + c_{37}z^3x + c_{38}z^2y^2 + c_{39}z^3y + c_{40}z^4, \\ F_6 &= x^2yz + c_{41}z^2x^2 + c_{42}y^4 + c_{43}z^2yx + c_{44}zy^3 + c_{45}z^3x + c_{46}z^2y^2 + c_{47}z^3y + c_{48}z^4, \\ F_7 &= xy^2z + c_{49}z^2x^2 + c_{50}y^4 + c_{51}z^2yx + c_{52}zy^3 + c_{53}z^3x + c_{54}z^2y^2 + c_{55}z^3y + c_{56}z^4, \\ F_8 &= y^5 + c_{57}z^3x^2 + c_{58}zy^4 + c_{59}z^3yx + c_{60}z^2y^3 + c_{61}z^4x + c_{62}z^3y^2 + c_{63}z^4y + c_{64}z^5, \end{aligned}$$

by Maple 12 we compute the ideal  $\mathfrak{R}$  and the following ideal defining the Zariski tangent space  $T$  to  $\mathcal{S}(J)$  at the origin that has dimension 16

$$\begin{aligned} I(T) &= (c_{64}, c_{63}, c_{61}, c_{56}, c_{55}, c_{53}, c_{48}, c_{47}, c_{46}, c_{45}, c_{44}, c_{40}, c_{39}, c_{38}, c_{37}, c_{36}, c_{32}, c_{31}, c_{30}, c_{29}, \\ &\quad c_{28} - c_{54}, c_{27}, c_{26} - c_{52}, c_{25}, c_{24}, c_{23}, c_{22}, c_{21}, c_{20}, c_{19}, c_{18}, c_{17}, c_{16}, c_{15}, c_{14}, c_{13}, \\ &\quad c_{12}, c_{11}, c_{10}, c_9, c_8, c_7, c_6, c_5, c_4, c_3, c_2, c_1). \end{aligned}$$

In the ideal  $\mathfrak{R}$  we eliminate several variables of type  $C$  by applying [12, Theorem 5.4] and by substituting variables that appear only in the linear part of some polynomials of  $\mathfrak{R}$ . It follows that  $\mathcal{S}(J)$  can be isomorphically projected on a linear space  $T' \simeq \mathbb{A}^{19}$  containing  $T$ . In this embedding,  $\mathcal{S}(J)$  is the complete intersection of the following three hypersurfaces in  $\mathbb{A}^{19}$  of degrees 4, 4 and 8, respectively:

$$\begin{aligned} G_1 &= c_{41}^2c_{49}c_{50} + c_{41}c_{49}c_{50}c_{51} + c_{41}c_{50}^2c_{57} + c_{42}c_{49}c_{50}c_{57} + c_{43}c_{49}^2c_{50} + c_{49}c_{50}^2c_{59} + c_{49}c_{50}c_{51}^2 + c_{50}^2c_{51}c_{57} + \\ &\quad c_{50}^2c_{57}c_{58} - c_{41}c_{49}c_{52} - c_{49}c_{50}c_{53} - c_{49}c_{51}c_{52} - 2c_{50}c_{52}c_{57} + c_{33}c_{49} - c_{41}^2 + c_{41}c_{51} - c_{42}c_{57} - c_{43}c_{49} + \\ &\quad c_{49}c_{54} - c_{53}, \\ G_2 &= c_{41}c_{42}c_{49}c_{50} + c_{42}c_{49}c_{50}c_{51} + c_{42}c_{49}c_{50}c_{58} + c_{42}c_{50}^2c_{57} + c_{43}c_{49}c_{50}^2 + c_{50}^3c_{59} + c_{50}^2c_{51}^2 + c_{50}^2c_{51}c_{58} + \\ &\quad c_{50}^2c_{58}^2 - c_{42}c_{49}c_{52} - c_{44}c_{49}c_{50} - c_{50}^2c_{53} - c_{50}^2c_{60} - 2c_{50}c_{51}c_{52} - 2c_{50}c_{52}c_{58} + c_{34}c_{49} - c_{41}c_{42} + c_{42}c_{51} - \\ &\quad c_{42}c_{58} - c_{43}c_{50} + 2c_{50}c_{54} + c_{52}^2 + c_{44}, \\ G_3 &= -c_{41}^3c_{49}c_{50}^2 - c_{41}^2c_{49}^3c_{50}^2c_{51} + c_{41}^2c_{49}^3c_{50}^2c_{58} - 2c_{41}^2c_{49}^2c_{50}^3c_{57} + c_{41}c_{42}^2c_{49}^5 + 2c_{41}c_{49}^3c_{50}^2c_{51}c_{58} + \\ &\quad c_{41}c_{49}^3c_{50}^2c_{58}^2 - 2c_{41}c_{49}^2c_{50}^3c_{51}c_{57} - c_{41}c_{49}^2c_{50}^4c_{57}^2 + c_{42}^2c_{49}^5c_{51} + c_{42}^2c_{49}^5c_{58} - c_{49}^3c_{50}^2c_{51}^2c_{58} - c_{49}^3c_{50}^2c_{58}^3 + \\ &\quad 2c_{49}^2c_{50}^3c_{51}c_{57}c_{58} + 2c_{49}^2c_{50}^3c_{57}c_{58}^2 - c_{49}c_{50}^4c_{51}c_{57}^2 - c_{49}c_{50}^4c_{57}^2c_{58} + 2c_{41}^2c_{49}^3c_{50}^2c_{52} - 2c_{41}c_{42}c_{49}^4c_{52} \end{aligned}$$

$$\begin{aligned}
& -4c_{41}c_{49}^3c_{50}c_{52}c_{58} + 4c_{41}c_{49}^2c_{50}^2c_{52}c_{57} - 2c_{42}c_{44}c_{49}^5 - 2c_{42}c_{49}^4c_{50}c_{60} - 2c_{42}c_{49}^4c_{51}c_{52} + 2c_{49}^3c_{50}c_{52}c_{58}^2 \\
& - 4c_{49}^2c_{50}^2c_{52}c_{57}c_{58} + 2c_{49}c_{50}^3c_{52}c_{57}^2 - 2c_{33}c_{41}c_{49}^3c_{50} + 2c_{33}c_{49}^3c_{50}c_{58} - 2c_{33}c_{49}^2c_{50}^2c_{57} + 2c_{34}c_{41}c_{49}^4 - \\
& 2c_{34}c_{49}^4c_{58} + 2c_{34}c_{49}^3c_{50}c_{57} + 4c_{41}^3c_{49}^2c_{50} - c_{41}^2c_{42}c_{49}^3 - 2c_{41}^2c_{49}^2c_{50}c_{51} - 4c_{41}^2c_{49}^2c_{50}c_{58} + 5c_{41}^2c_{49}c_{50}^2c_{57} + \\
& 3c_{41}c_{42}c_{49}^3c_{51} + 4c_{41}c_{42}c_{49}^2c_{50}c_{57} + 3c_{41}c_{43}c_{49}^3c_{50} + c_{41}c_{49}^3c_{52}^2 + c_{41}c_{49}^2c_{50}^2c_{59} + c_{41}c_{49}c_{50}^2c_{51}c_{57} + 2c_{41}c_{50}^3c_{57}^2 + \\
& c_{42}c_{43}c_{49}^4 + 2c_{42}c_{49}^4c_{54} + 3c_{42}c_{49}^3c_{50}c_{59} + c_{42}c_{49}^3c_{51}^2 - c_{42}c_{49}^3c_{51}c_{58} + c_{42}c_{49}^3c_{58}^2 - 2c_{42}c_{49}^2c_{50}c_{51}c_{57} - \\
& 4c_{42}c_{49}^2c_{50}c_{57}c_{58} + 2c_{42}c_{49}c_{50}^2c_{57}^2 - 3c_{43}c_{49}^3c_{50}c_{58} + 3c_{43}c_{49}^2c_{50}^2c_{57} + 2c_{44}c_{49}^4c_{52} + 2c_{49}^3c_{50}c_{52}c_{60} + c_{49}^3c_{51}c_{52}^2 - \\
& c_{49}^3c_{52}^2c_{58} - c_{49}^2c_{50}^2c_{58}c_{59} - c_{49}^2c_{50}^2c_{51}^2 - 2c_{49}^2c_{50}^2c_{51}c_{58} + c_{49}c_{50}^3c_{57}c_{59} - c_{49}c_{50}^2c_{51}^2c_{57} - 5c_{49}c_{50}^2c_{51}c_{57}c_{58} - \\
& 3c_{49}c_{50}^2c_{57}c_{58}^2 + 2c_{50}^3c_{51}c_{57}^2 + 2c_{50}^3c_{57}^2c_{58} - c_{41}c_{49}^2c_{52} + c_{41}c_{44}c_{49}^3 + c_{41}c_{49}^2c_{50}c_{60} + c_{41}c_{49}^2c_{51}c_{52} + c_{41}c_{49}^2c_{52}c_{58} - \\
& 5c_{41}c_{49}c_{50}c_{52}c_{57} - 2c_{42}c_{49}^3c_{53} + c_{42}c_{49}^3c_{60} + c_{42}c_{49}^2c_{52}c_{57} + c_{43}c_{49}^3c_{52} - 2c_{44}c_{49}^3c_{51} - c_{44}c_{49}^3c_{58} - 2c_{49}^3c_{52}c_{54} + \\
& c_{49}^2c_{50}c_{51}c_{60} - c_{49}^2c_{50}c_{52}c_{59} + 2c_{49}^2c_{51}^2c_{52} + c_{49}^2c_{51}c_{52}c_{58} + 2c_{49}c_{50}^2c_{57}c_{60} + 5c_{49}c_{50}c_{51}c_{52}c_{57} + 6c_{49}c_{50}c_{52}c_{57}c_{58} - \\
& 4c_{50}^2c_{52}c_{57} + c_{33}c_{41}c_{49}^2 - 2c_{33}c_{49}^2c_{51} - c_{33}c_{49}^2c_{58} + c_{33}c_{49}c_{50}c_{57} - 3c_{34}c_{49}^2c_{57} - c_{35}c_{49}^3 - 2c_{41}^3c_{49} + 2c_{41}^2c_{49}c_{51} + \\
& 2c_{41}^2c_{49}c_{58} - 3c_{41}^2c_{50}c_{57} - c_{41}c_{42}c_{49}c_{57} - 2c_{41}c_{43}c_{49}^2 + c_{41}c_{49}^2c_{54} - 2c_{41}c_{49}c_{50}c_{59} - 3c_{41}c_{49}c_{51}^2 + c_{41}c_{50}c_{51}c_{57} - \\
& c_{42}c_{49}^2c_{59} - c_{42}c_{49}c_{51}c_{57} + 3c_{42}c_{49}c_{57}c_{58} - 3c_{42}c_{50}c_{57}^2 + 2c_{43}c_{49}^2c_{58} - 2c_{43}c_{49}c_{50}c_{57} - c_{49}^2c_{50}c_{62} - 2c_{49}^2c_{51}c_{54} - \\
& c_{49}^2c_{54}c_{58} - 2c_{49}c_{50}c_{51}c_{59} - c_{49}c_{50}c_{54}c_{57} - c_{49}c_{51}^3 + c_{49}c_{51}^2c_{58} - 2c_{49}c_{52}^2c_{57} - c_{50}^2c_{57}c_{59} - c_{50}c_{51}^2c_{57} - \\
& c_{41}c_{49}c_{60} + c_{41}c_{52}c_{57} - c_{44}c_{49}c_{57} + 2c_{49}c_{50}c_{61} + 4c_{49}c_{51}c_{53} - c_{49}c_{51}c_{60} + c_{49}c_{52}c_{59} - c_{50}c_{53}c_{57} + c_{51}c_{52}c_{57} + \\
& c_{33}c_{57} + c_{41}c_{59} + c_{49}c_{62} - c_{54}c_{57} - c_{61}.
\end{aligned}$$

Among the generators of the corresponding Jacobian ideal we have the following minors  $D_i$ 's obtained by computing derivatives of  $G_1, G_2, G_3$  with respect to the sets of variables  $A_i$ 's, for  $1 \leq i \leq 5$ :

$$\begin{aligned}
D_1 &:= -(2c_{49}c_{50} - 1)(c_{49}c_{50} - 1)(c_{49}c_{50} + 1), \quad A_1 = \{c_{61}, c_{44}, c_{53}\}; \\
D_2 &= -(c_{49}c_{50} + 1)(c_{49}c_{50} - 1)^2c_{49}, \quad A_2 = \{c_{53}, c_{44}, c_{62}\}; \\
D_3 &= -c_{50}(2c_{49}c_{50} - 1)(c_{49}c_{50} - 1), \quad A_3 = \{c_{43}, c_{61}, c_{53}\}; \\
D_4 &= c_{49}(c_{49}c_{50} - 1)^2(2c_{49}c_{50} - 1), \quad A_4 = \{c_{43}, c_{61}, c_{44}\}; \\
D_5 &= (c_{49}c_{50} + 1)c_{50}^2(2c_{49}c_{50} - 1), \quad A_5 = \{c_{53}, c_{60}, c_{61}\}.
\end{aligned}$$

The polynomials  $D_i$ 's define the empty set, so that  $S(J)$  is smooth as we expected and, in particular,  $J$  corresponds to a smooth point on  $S(J)$ . Moreover,  $\mathcal{Mf}(J)$  has dimension 16 but we claim that it cannot be isomorphically projected on  $T$ . Indeed, note that we can choose a set of 16 variables that is complementary to the tangent space and that do not contain the variables  $c_{53}, c_{44}, c_{61}$  which occur in the linear parts of the polynomials  $G_i$ 's. These variables appear also in other parts of the polynomials and their coefficients are  $c_{49}c_{50} + 1$ ,  $c_{49}c_{50} - 1$  and  $2c_{49}c_{50} - 1$ , respectively. If  $\bar{c} \in \mathbb{T}$  is a point of the tangent space on which none of the coefficients vanishes, we obtain a unique point of  $\mathcal{Mf}(J)$  of which  $\bar{c}$  is the projection on  $T$ . If  $\bar{c} \in T$  is a general point of the tangent space on which one of this coefficient vanishes, one can see that  $\bar{c}$  is not the projection of any point of  $\mathcal{Mf}(J)$ . Hence, the projection of  $\mathcal{Mf}(J)$  on  $T$  does not coincide with the tangent space  $T$ , but only with an open set. However, this fact implies that  $\mathcal{Mf}(J)$  is rational, in particular irreducible.

We point out that the variables  $c_{49}$  and  $c_{50}$ , that appear in the coefficients of the variables  $c_{53}, c_{44}, c_{61}$ , are the coefficients in the polynomial  $F_7$  of the two terms  $z^2x^2, y_4$  whose behaviour prevents the ideal  $J$  from being a segment. Indeed, in this case the affine scheme  $\mathcal{Mf}(J)$  is homogeneous with respect to a non-positive grading.

## REFERENCES

- [1] Michael Artin, *Lectures on deformations of singularities*, Tata Institute on Fundamental Research, Bombay, 1976, Notes by C. S. Seshadri and Allen Tannenbaum.

- [2] Dave Bayer and David Mumford, *What can be computed in algebraic geometry?*, Computational algebraic geometry and commutative algebra (Cortona, 1991), Sympos. Math., XXXIV, Cambridge Univ. Press, Cambridge, 1993, pp. 1–48.
- [3] Bruno Buchberger, *Gröbner bases - an algorithmic method in polynomial ideal theory*, Multidimensional systems Theory (N. K. Bose, ed.), Reidel Publishing Company, 1985, pp. 184–232.
- [4] Giuseppa Carrà Ferro, *Gröbner bases and Hilbert schemes. I*, J. Symbolic Comput. **6** (1988), no. 2-3, 219–230, Computational aspects of commutative algebra.
- [5] Todd Deery, *Rev-lex segment ideals and minimal Betti numbers*, The Curves Seminar at Queen’s, Vol. X (Kingston, ON, 1995), Queen’s Papers in Pure and Appl. Math., vol. 102, Queen’s Univ., Kingston, ON, 1996, pp. 193–219.
- [6] Giorgio Ferrarese and Margherita Roggero, *Homogeneous varieties for Hilbert schemes*, Int. J. Algebra **3** (2009), no. 9-12, 547–557.
- [7] André Galligo, *Théorème de division et stabilité en géométrie analytique locale*, Ann. Inst. Fourier (Grenoble) **29** (1979), no. 2, vii, 107–184.
- [8] Rüdiger Gebauer and H. Michael Möller, *Computation of minimal generators of ideals of fat points*, SYM-SAC ’86 Proceedings of the fifth ACM symposium on Symbolic and algebraic computation (New York), ACM, 1986, pp. 218–221 (electronic).
- [9] Robin Hartshorne, *Algebraic geometry*, Springer-Verlag, New York, 1977, Graduate Texts in Mathematics, No. 52.
- [10] Martin Kreuzer and Lorenzo Robbiano, *Computational commutative algebra. 1*, Springer-Verlag, Berlin, 2000.
- [11] Paolo Lella and Margherita Roggero, *Rational components of Hilbert schemes*, Available at arXiv:0903.1029, 2009.
- [12] ———, *Borel open coverings of Hilbert schemes*, Preprint, 2010.
- [13] Tie Luo and Erol Yilmaz, *On the lifting problem for homogeneous ideals*, J. Pure Appl. Algebra **162** (2001), no. 2-3, 327–335.
- [14] H. Michael Möller and Ferdinando Mora, *New constructive methods in classical ideal theory*, J. Algebra **100** (1986), no. 1, 138–178.
- [15] H. Michael Möller, Ferdinando Mora, and Carlo Traverso, *Gröbner Bases Computation Using Syzygies*, Proceedings of the 1992 International Symposium on Symbolic and Algebraic Computation (New York), ACM, 1992, pp. 320–328 (electronic).
- [16] R. Notari and M. L. Spreafico, *A stratification of Hilbert schemes by initial ideals and applications*, Manuscripta Math. **101** (2000), no. 4, 429–448.
- [17] Alyson Reeves and Bernd Sturmfels, *A note on polynomial reduction*, J. Symbolic Comput. **16** (1993), no. 3, 273–277.
- [18] Lorenzo Robbiano, *On border basis and Gröbner basis schemes*, Collect. Math. **60** (2009), no. 1, 11–25.
- [19] Margherita Roggero and Lea Terracini, *Ideals with an assigned initial ideal*, International Mathematical Forum **5** (2010), no. 55, 2731–2750.
- [20] Giuseppe Valla, *Problems and results on Hilbert functions of graded algebras*, Six lectures on commutative algebra (Bellaterra, 1996), Progr. Math., vol. 166, Birkhäuser, Basel, 1998, pp. 293–344.

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